Aiyagari Model with Labor Supply: Endogenous Grid Method

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The goal of this note is to illustrate how to solve a standard incomplete market models (a la Aiyagari-Hugget) with labor supply using the Endogenous Grid Method (EGM).

1 Model Setup

$$
\begin{aligned}
V(a,s) &= \max_{c,h,a'} u(c) - \nu(h) + \beta \mathbb{E}\big[V(a',s')|s\big] \\
s.t.\quad & c + a' = wsh + (1+r)a \\
& \log s_t = \rho \log s_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0,\sigma^2) \\
& a' \geq -b, \quad h \geq 0, \quad c \geq 0
\end{aligned}
$$

Derive the household's optimality conditions with respect to consumption, labor supply, and future asset holdings. Note that we can ignore the positivity constraints on consumption and labor supply, the functional forms of utility will make sure later this is satisfied. But we have to incorporate the borrowing constraint on the asset holdings.

Bellman equation:

$$V(a,s) = \max_{c,h,a'} u(c) - \nu(h) + \beta \mathbb{E}\big[V(a',s')|s\big] + \lambda\Big[wsh + (1+r)a - a' - c\Big] + \mu(b+a') \tag{1}$$

Intratemporal optimality condition

$$\nu'(h) = u'(c)sw \tag{2}$$

Intertemporal optimality condition: E.E.

$$u'(c) = \beta(1+r)\mathbb{E}\big[u'(c')\big] + \mu \tag{3}$$

KTT condition

$$\mu(b+a') = 0, \quad \mu \geq 0 \tag{4}$$

Budget constraint

$$c + a' = wsh + (1+r)a \tag{5}$$

Equations (2)-(5) characterize the optimality of the household's choice.

2 Algorithm Summary

In EGM, we guess $c'(a', s')$ on the future-asset grid a' for each s'. Then

- Let the inverse function of the marginal disutility of labor be denoted by $(\nu')^{-1}(\cdot)$. Using the intratemporal optimality condition, the optimal labor supply is a function of the optimal consumption and the productivity realization

$$\mathcal{H}(c, s) = (\nu')^{-1}(u'(c)sw)$$

- Guess a decision rule for the optimal consumption as a function of the future states,

$$c_0(a', s')$$

  Let the future asset level be consistent with a non-binding borrowing constraint, $a' \geq -b$.

- Denote the inverse marginal utility function of consumption by $(u')^{-1}(\cdot)$. Use Euler to back out current $C_0(a', s)$ for each pair (a', s). Hence, $C_0(a', s)$ measures the current consumption level as a function of the future asset a', the current productivity s, and the guess for future consumption $c_0(a', s')$ if the borrowing constraint is NOT binding: $\mu = 0$.

$$\begin{aligned} C_0(a', s) &= (u')^{-1}\Big(\beta(1+r)\mathbb{E}\big[u'(c_0(a', s'))\big]\Big) \\ &= (u')^{-1}\Big(\beta(1+r)\sum_{s'}\pi(s, s')u'(c_0(a', s'))\Big) \end{aligned}$$

- Convert $(c(a', s), a')$ into the endogenous current asset using the budget constraint.
  $\Rightarrow$ Find the current asset level $\mathcal{A}_0(a', s)$ that is consistent with the guess of the consumption policy. $\mathcal{A}_0(a', s)$ is the current assets holdings for a household with labor productivity s who choose to save a'.

$$\mathcal{A}_0(a', s) = \frac{C_0(a', s) + a' - ws\mathcal{H}(C_0(a', s), s)}{1+r} \equiv a_{endo}(a', s)$$

- The pair of functions, $< C_0(a', s), \mathcal{A}_0(a', s) >$, define a relationship between current consumption c, current asset level a, and productivity realization s: $(a, s) \to c$, given that the borrowing constraint does not bind: $\mu = 0$. Hence, the guess on the consumption function can be updated by interpolating $c_0(\mathcal{A}_0(a', s), s)$ on (a', s') to get

$$c_1(a', s'), \quad a' \geq \mathcal{A}_0(-b, s)$$

- Now we need to think about the binding case.
  We update the consumption function that is consistent with a binding borrowing constraint $a' = -b$, for future asset levels a' that are smaller than the lowest current asset level that is exactly consistent with a binding borrowing constraint in the future

$$a' < \mathcal{A}_0(-b, s)$$

- Check the difference between $c_0(a', s')$ and $c_1(a', s')$